2nd International Conference on Pioneer and Innovative Studies

January 30-31, 2025 : Konya, Turkey

© 2025 Published by All Sciences Academy

https://as-proceeding.com/index.php/icpis/home

Pedal Curves and Applications According to a Vector lying on the Osculator Plane

SÜLEYMAN ŞENYURT[1] and FİLİZ ERTEM KAYA[2,*]

[1]Department of Mathematics/Faculty of Science-Art, Ordu University, TÜRKİYE
[2]Department of Mathematics/ Faculty of Science, Niğde Ömer Halisdemir University, TÜRKİYE

[*](fertem@ohu.edu.tr) Email of the corresponding author

Abstract – In this study, the pedal curve of a unit vector lying on the osculator plane of a curve were defined and the Frenet vectors, curvature, and torsion of these pedal curves were calculated. Then, the Smarandache curves of these pedal curves were defined by taking the position vector of the Frenet vectors of the pedal curve. Finally, for each Smarandache curve, the expressions of the Frenet vectors, curvatures and torsions depending on the main curves were obtained.

Keywords – Pedal Curves, Smarandache Curves, Frenet Vectors, Curvature, Torsion 2010 AMS Classification: 53A04

I. INTRODUCTION

The curves are natural figures that always exist in nature. Scientists have formulated these figures, examined their characteristics features and obtained some theories. Curve theory has become one of the most representative and most studied areas of differential geometry.

When an orthonormal system is established on a curve, more information and theory can be obtained. Any two curves can be connected to each other. For example, involute-evolute curves, Bertrand and Mannheim curves, successor curves, Smarandache curves are such curves. There are many studies are such curves, (e.g. [1], [7], [13], [14]).

The geometric locus of the perpendicular projection points from a point not on the curve onto the tangent and normal vector of a curve is defined as a pedal (or contrapedal) curve. Studies on such curves are given in the sources, (e.g. [3–5], [8–12], [15–17]). Studies on curves have been calculated by taking different frames in many spaces and similar studies also continue today.

In this study, the pedal curve of a unit vector lying on the osculator plane of a curve was defined and the Frenet vectors, curvature and torsion were calculated. Then, Smarandache curves were described by taking Frenet vectors of these pedal curves as position vectors. Finally, Frenet vectors, curvature and torsion of the obtained curves were expressed in terms of the main curve.

Frenet vectors, curvatures and Frenet formulae of a differentiable curve $\alpha(t)$ are given as following equalities, respectively.

$$T=\frac{\alpha'}{\|\alpha'\|}, N=B\wedge T,\ \ B=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|}, \tag{1.1}$$

$$\kappa=\frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3},\ \ \tau=\frac{\det(\alpha',\alpha'',\alpha''')}{\|\alpha'\wedge\alpha''\|^2}, \tag{1.2}$$

$$T'=v\kappa N,\ \ N'=v(-\kappa T+\tau B),\ B'=-v\tau N \tag{1.3}$$

where $v=\|\alpha'\|$, (e.g. [2], [5]). The spaces $sp\{T,N\}$, $sp\{N,B\}$, $sp\{T,B\}$ obtained from the Frenet vectors are called osculating, normal and rectifying planes, respectively. The unit vector lying on the osculator plane can be taken as, $X=\dfrac{xT+yN}{\sqrt{x^2+y^2}}$, $x,y\in IR$.

Now, let's give some characterizations of the $X-pedal$ curve that are the geometric locus of the perpendicular projection points to onto this vector X.

II. PEDAL CURVES AND APPLICATIONS ACCORDING TO A VECTOR LYING ON THE OSCULATOR PLANE

Definiton II.1. Let X be the unit vector lying on the osculator plane of a regular curve α in Euclidean 3-Space E^3. The geometric locus of the perpendicular projection points onto the vector X from a point $P\in E^3$ that is not on the curve is called the $X-pedal$ curve according to the point P of curve α. The equation of $X-pedal$ curve is written as the following equality.

$$\alpha_X(t)=\alpha(t)+u(t)X(t) \qquad (2.1)$$

[15-17]. Specifically, if the point olarak P is origin, then we have $u(t)=-\langle\alpha(t),X(t)\rangle$. For convenience in operations, the equation of the vector X will be taken as the following equation.

$$X=\frac{1}{\sqrt{2}}(T+N)$$

Theorem II.1. Let α_X be the $X-pedal$ curve of a curve α with unit speed and T_1, N_1 and B_1 be the Frenet vectors of the curve α_X, respectively. These vectors are given as follows.

$$T_1=\omega_1\left(\sqrt{2}+u'-u\kappa\right)T+\omega_1(u'+u\kappa)N+\omega_1(u\tau)B$$

$$N_1=B_1\wedge T_1$$

$$\begin{aligned}B_1=&\eta_1\Big((u'+u\kappa)(2\tau u'+u\tau'+u\tau\kappa)-u\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\right)\Big)T\\&+\eta_1\Big(u\tau\left((u'-u\kappa)'-\kappa(u'+u\kappa)\right)-\left(\sqrt{2}+u'-u\kappa\right)(2\tau u'+u\tau'+u\tau\kappa)\Big)N\\&+\eta_1\Big(\left(\sqrt{2}+u'-u\kappa\right)\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\right)-(u'+u\kappa)\left((u'-u\kappa)'-\kappa(u'+u\kappa)\right)\Big)B\end{aligned}$$

where

$$\omega_1=\frac{1}{\sqrt{\left(\sqrt{x^2+y^2}+u'x-uy\kappa\right)^2+\left(u'y+ux\kappa\right)^2+\left(uy\tau\right)^2}},$$

$$\eta_1=\frac{1}{\left(\begin{array}{l}\Big((u'+u\kappa)(2\tau u'+u\tau'+u\tau\kappa)-u\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\right)\Big)^2\\+\Big(u\tau\left((u'-u\kappa)'-\kappa(u'+u\kappa)\right)-\left(\sqrt{2}+u'-u\kappa\right)(2\tau u'+u\tau'+u\tau\kappa)\Big)^2\\+\Big(\left(\sqrt{2}+u'-u\kappa\right)\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\right)-(u'+u\kappa)\left((u'-u\kappa)'-\kappa(u'+u\kappa)\right)\Big)^2\end{array}\right)^{\frac{1}{2}}}.$$

Proof. If the derivatives of (2.1) are taken and necessary algebraic operations are performed, then we take following equations.

$$\alpha'_X = \frac{\left(\sqrt{2}+u'-u\kappa\right)T+\left(u'+u\kappa\right)N+u\tau B}{\sqrt{2}} \tag{2.2}$$

$$\alpha''_X = \frac{1}{\sqrt{2}}\left(\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)T+\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)N+\left(2\tau u'+u\tau'+u\tau\kappa\right)B\right) \tag{2.3}$$

$$\alpha'''_X = \frac{1}{\sqrt{2}}\begin{pmatrix}\left(\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)'-\kappa\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)T\\ +\left(\kappa\left(u'-u\kappa\right)'-\kappa^2\left(u'+u\kappa\right)-\left(2\tau^2u'+u\tau\tau'+u\tau^2\kappa\right)+\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)'\right)N\\ +\left(\left(2\tau u'+u\tau'+u\tau\kappa\right)'+\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)B\end{pmatrix} \tag{2.4}$$

$$\alpha'_X\wedge\alpha''_X = \frac{\begin{pmatrix}\left(u'+u\kappa\right)\left(2\tau u'+u\tau'+u\tau\kappa\right)-u\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)T\\ +\left(u\tau\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)-\left(\sqrt{2}+u'-u\kappa\right)\left(2\tau u'+u\tau'+u\tau\kappa\right)\right)N\\ +\left(\left(\sqrt{2}+u'-u\kappa\right)\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)-\left(u'+u\kappa\right)\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)\right)B\end{pmatrix}}{2} \tag{2.5}$$

$$\det\left(\alpha'_X,\alpha''_X,\alpha'''_X\right) = \frac{1}{2\sqrt{2}}\begin{pmatrix}\left(\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)'-\kappa\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right).\\ \left(\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)'-\kappa\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)\\ +\left(u\tau\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)-\left(\sqrt{2}+u'-u\kappa\right)\left(2\tau u'+u\tau'+u\tau\kappa\right)\right).\\ \left(\kappa\left(u'-u\kappa\right)'-\kappa^2\left(u'+u\kappa\right)-\left(2\tau^2u'+u\tau\tau'+u\tau^2\kappa\right)+\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)'\right)\\ +\left(\left(\sqrt{2}+u'-u\kappa\right)\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)-\left(u'+u\kappa\right)\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)\right).\\ \left(\left(2\tau u'+u\tau'+u\tau\kappa\right)'+\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)\end{pmatrix} \tag{2.6}$$

$$\left\|\alpha'_X\right\| = \frac{\sqrt{\left(\sqrt{2}+u'-u\kappa\right)^2+\left(u'+u\kappa\right)^2+\left(u\tau\right)^2}}{\sqrt{2}}, \tag{2.7}$$

$$\left\|\alpha'_X\wedge\alpha''_X\right\| = \frac{1}{2}\begin{pmatrix}\left(\left(u'+u\kappa\right)\left(2\tau u'+u\tau'+u\tau\kappa\right)-u\tau\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)\right)^2\\ +\left(u\tau\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)-\left(\sqrt{2}+u'-u\kappa\right)\left(2\tau u'+u\tau'+u\tau\kappa\right)\right)^2\\ +\left(\left(\sqrt{2}+u'-u\kappa\right)\left(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+\left(u'+u\kappa\right)'\right)-\left(u'+u\kappa\right)\left(\left(u'-u\kappa\right)'-\kappa\left(u'+u\kappa\right)\right)\right)^2\end{pmatrix}^{\frac{1}{2}} \tag{2.8}$$

The proof is done upon substituting the above expressions $(2.2),(2.5),(2.7)$ and (2.8) into the relation (1.1).

Theorem II.2. Let κ_1 and τ_1 denote the curvature and torsion of the $X-pedal$ curve α_X of a unit speed curve α, respectively. These curvatures are obtained as follows.

$$\kappa_1=\frac{\left(\begin{array}{l}2\Big((u'+u\kappa)(2\tau u'+u\tau'+u\tau\kappa)-u\tau\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)\Big)^2\\+2\Big(u\tau\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)-\left(\sqrt{2}+u'-u\kappa\right)(2\tau u'+u\tau'+u\tau\kappa)\Big)^2\\+2\Big(\left(\sqrt{2}+u'-u\kappa\right)\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)-(u'+u\kappa)\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)\Big)^2\end{array}\right)^{\frac{1}{2}}}{\left(\left(\sqrt{2}+u'-u\kappa\right)^2+(u'+u\kappa)^2+(u\tau)^2\right)^{\frac{3}{2}}},$$

$$\tau_1=\frac{\sqrt{2}\left(\begin{array}{l}\Big(\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)'-\kappa\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)\Big).\\\Big(\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)'-\kappa\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)\Big)\\+\Big(u\tau\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)-\left(\sqrt{2}+u'-u\kappa\right)(2\tau u'+u\tau'+u\tau\kappa)\Big).\\\Big(\kappa(u'-u\kappa)'-\kappa^2(u'+u\kappa)-\left(2\tau^2u'+u\tau\tau'+u\tau^2\kappa\right)+\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)'\Big)\\+\Big(\left(\sqrt{2}+u'-u\kappa\right)\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)-(u'+u\kappa)\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)\Big).\\\Big((2\tau u'+u\tau'+u\tau\kappa)'+\tau\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)\Big)\end{array}\right)}{\left(\begin{array}{l}\Big((u'+u\kappa)(2\tau u'+u\tau'+u\tau\kappa)-u\tau\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)\Big)^2\\+\Big(u\tau\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)-\left(\sqrt{2}+u'-u\kappa\right)(2\tau u'+u\tau'+u\tau\kappa)\Big)^2\\+\Big(\left(\sqrt{2}+u'-u\kappa\right)\Big(\sqrt{2}\kappa+u'\kappa-u\left(\kappa^2+\tau^2\right)+(u'+u\kappa)'\Big)-(u'+u\kappa)\Big((u'-u\kappa)'-\kappa(u'+u\kappa)\Big)\Big)^2\end{array}\right)}.$$

Proof. By substituting equalities $(2.6),(2.7)$ and (2.8) into the relations at (1.2), the proof is completed.

Corollary II.1. The following relations exist between the Frenet vectors of the $X-pedal$ curve and their derivatives:

$$T_1'=\mu_1\kappa_1N_1,\ \ N_1'=\mu_1\left(-\kappa_1T_1+\tau_1B_1\right),\ B_1'=-\mu_1\tau_1N_1 \qquad (2.9)$$

where $\mu_1=\left\|\alpha_X'\right\|$.

Definiton II.2. By taking the tangent and the principal normal vectors of the $X-pedal$ curve as position vectors, we define a regular curve called the α_1- Smarandache curve belonging to $X-pedal$ curve obtained from these vectors.

$$\alpha_1=\frac{T_1+N_1}{\sqrt{2}} \tag{2.10}$$

Theorem II.3. If the Frenet vectors of the α_1- Smarandache curve are shown as T_{α_1}, N_{α_1} and B_{α_1}, then these vectors are obtained as follows.

$$T_{\alpha_1}=\frac{-\kappa_1T_1+\kappa_1N_1+\tau_1B_1}{\sqrt{2\kappa_1^2+\tau_1^2}}$$

$$N_{\alpha_1}=B_{\alpha_1}\wedge T_{\alpha_1}$$

$$B_{\alpha_1}=\frac{-\left(\kappa_1x_3+\tau_1x_2\right)T_1+\left(\kappa_1x_3+\tau_1x_1\right)N_1-\left(\kappa_1x_2+\kappa_1x_1\right)B_1}{\sqrt{\left(\kappa_1x_3+\tau_1x_2\right)^2+\left(\kappa_1x_3+\tau_1x_1\right)^2+\left(\kappa_1x_2+\kappa_1x_1\right)^2}}$$

where x_1,x_2 and x_3,

$$x_1=-\frac{\mu_1^2\kappa_1^2+\left(\mu_1\kappa_1\right)'}{\sqrt{2}},x_2=\frac{\left(\mu_1\kappa_1\right)'-\mu_1^2\left(\kappa_1^2+\tau_1^2\right)}{\sqrt{2}},x_3=\frac{\mu_1^2\kappa_1\tau_1+\left(\mu_1\tau_1\right)'}{\sqrt{2}}.$$

Proof. By taking the derivative of the expression (2.10) and then, taking into account the relation (2.9), we take the following equalities.

$$\alpha_1'=\frac{\mu_1\left(-\kappa_1T_1+\kappa_1N_1+\tau_1B_1\right)}{\sqrt{2}} \tag{2.11}$$

$$\alpha_1''=x_1T_1+x_2N_1+x_3B_1 \tag{2.12}$$

$$\alpha_1'''=\left(x_1'-\mu_1\kappa_1x_2\right)T_1+\left(x_2'+\mu_1x_1-\mu_1x_3\right)N_1+\left(x_3'+\mu_1\tau_1x_2\right)B_1 \tag{2.13}$$

$$\alpha_1'\wedge\alpha_1''=-\frac{\mu_1\left(\kappa_1x_3+\tau_1x_2\right)}{\sqrt{2}}T_1+\frac{\mu_1\left(\kappa_1x_3+\tau_1x_1\right)}{\sqrt{2}}N_1-\frac{\mu_1\left(\kappa_1x_2+\kappa_1x_1\right)}{\sqrt{2}}B_1 \tag{2.14}$$

$$\det\left(\alpha_1',\alpha_1'',\alpha_1'''\right)=\frac{\mu_1}{\sqrt{2}}\begin{pmatrix}\left(\kappa_1x_3+\tau_1x_1\right)\left(x_2'+\mu_1x_1-\mu_1x_3\right)\\-\left(\kappa_1x_3+\tau_1x_2\right)\left(x_1'-\mu_1\kappa_1x_2\right)\\-\left(\kappa_1x_2+\kappa_1x_1\right)\left(x_3'+\mu_1\tau_1x_2\right)\end{pmatrix} \tag{2.15}$$

$$\left\|\alpha_1'\right\|=\frac{\mu_1}{\sqrt{2}}\sqrt{2\kappa_1^2+\tau_1^2} \tag{2.16}$$

$$\left\|\alpha_1'\wedge\alpha_1''\right\|=\frac{\mu_1}{\sqrt{2}}\sqrt{\left(\kappa_1x_3+\tau_1x_2\right)^2+\left(\kappa_1x_3+\tau_1x_1\right)^2+\left(\kappa_1x_2+\kappa_1x_1\right)^2} \tag{2.17}$$

Substituting the relations (2.11), (2.14), (2.16) and (2.17) into (1.1) completes the proof.

Theorem II.4. Let κ_{α_1} and τ_{α_1} be the curvature and torsion of α_1- Smarandache curve, respectively. These curvatures are given by the following equalities.

$$\kappa_{\alpha_1}=\frac{2\sqrt{\left(\kappa_1x_3+\tau_1x_2\right)^2+\left(\kappa_1x_3+\tau_1x_1\right)^2+\left(\kappa_1x_2+\kappa_1x_1\right)^2}}{\mu_1^2\left(2\kappa_1^2+\tau_1^2\right)\sqrt{2\kappa_1^2+\tau_1^2}},$$

$$\tau_{\alpha_1}=\frac{\sqrt{2}\left(\left(\kappa_1x_3+\tau_1x_1\right)\left(x_2'+\mu_1x_1-\mu_1x_3\right)-\left(\kappa_1x_3+\tau_1x_2\right)\left(x_1'-\mu_1\kappa_1x_2\right)-\left(\kappa_1x_2+\kappa_1x_1\right)\left(x_3'+\mu_1\tau_1x_2\right)\right)}{\mu_1\left(\left(\kappa_1x_3+\tau_1x_2\right)^2+\left(\kappa_1x_3+\tau_1x_1\right)^2+\left(\kappa_1x_2+\kappa_1x_1\right)^2\right)}.$$

Proof. If substituting $(2.15),(2.16)$ and (2.17) into (1.2), then the proof is completed.

Definition II.3. By taking the tangent and binormal vectors of the $X-pedal$ curve as position vectors, we define a regular curve called the α_2- Smarandache curve belonging to $X-pedal$ curve obtained from these vectors.

$$\alpha_2=\frac{T_1+B_1}{\sqrt{2}}. \qquad (2.18)$$

Theorem II.5. If $T_{\alpha_2},N_{\alpha_2}$ and B_{α_2} denote the Frenet vectors of α_2- Smarandache curve, then we have the following relations.

$$T_{\alpha_2}=N_1,N_{\alpha_2}=\frac{-\kappa_1T_1+\tau_1B_1}{\sqrt{\kappa_1^2+\tau_1^2}},B_{\alpha_2}=\frac{\tau_1T_1+\kappa_1B_1}{\sqrt{\kappa_1^2+\tau_1^2}}$$

Proof. By taking the derivatives of (2.18) and then, taking the account of (2.9), following equalities are obtained.

$$\alpha_2'=\frac{\mu_1\left(\kappa_1-\tau_1\right)N_1}{\sqrt{2}} \qquad (2.19)$$

$$\alpha_2''=\frac{-\kappa_1\mu_1^2\left(\kappa_1-\tau_1\right)T_1+\left(\mu_1\kappa_1-\mu_1\tau_1\right)'N_1+\tau_1\mu_1^2\left(\kappa_1-\tau_1\right)B_1}{\sqrt{2}} \qquad (2.20)$$

$$\alpha_2'''=\frac{\begin{array}{l}\left(\left(-\kappa_1\mu_1^2\left(\kappa_1-\tau_1\right)\right)'-\kappa_1\mu_1\left(\mu_1\kappa_1-\mu_1\tau_1\right)''\right)T_1\\+\left(\left(\mu_1\kappa_1-\mu_1\tau_1\right)''-\mu_1^3\left(\kappa_1-\tau_1\right)\left(\kappa_1^2+\tau_1^2\right)\right)N_1\\+\left(\left(\tau_1\mu_1^2\left(\kappa_1-\tau_1\right)\right)'+\tau_1\mu_1\left(\mu_1\kappa_1-\mu_1\tau_1\right)''\right)B_1\end{array}}{\sqrt{2}} \qquad (2.21)$$

$$\alpha_2'\wedge\alpha_2''=\frac{\mu_1^3\left(\kappa_1-\tau_1\right)^2\tau_1T_1+\mu_1^3\left(\kappa_1-\tau_1\right)^2\kappa_1B_1}{2} \qquad (2.22)$$

$$\det\left(\alpha_2',\alpha_2'',\alpha_2'''\right)=\frac{\mu_1^5\left(\kappa_1-\tau_1\right)^3\left(\kappa_1\tau_1'-\tau_1\kappa_1'\right)}{2\sqrt{2}} \qquad (2.23)$$

$$\|\alpha_2'\| = \frac{\mu_1(\kappa_1 - \tau_1)}{\sqrt{2}} \tag{2.24}$$

$$\|\alpha_2' \wedge \alpha_2''\| = \frac{\mu_1^3(\kappa_1 - \tau_1)^2\sqrt{\kappa_1^2 + \tau_1^2}}{2} \tag{2.25}$$

When substituting (2.19), (2.22), (2.24) and (2.25) into (1.1), it completes the proof.

Theorem II.6. Let κ_{α_2} and τ_{α_2} be the curvature and torsion of α_2- Smarandache curve. The relations among the curvatures are given as follows.

$$\kappa_{\alpha_2} = \frac{\sqrt{2\kappa_1^2 + 2\tau_1^2}}{(\kappa_1 - \tau_1)}, \quad \tau_{\alpha_2} = \frac{\sqrt{2}(\kappa_1\tau_1' - \tau_1\kappa_1')}{\mu_1(\kappa_1 - \tau_1)(\kappa_1^2 + \tau_1^2)}.$$

Proof. The proof is clear by the substitution of (2.23), (2.24) and (2.25) into the relation (1.2).

Definiton II.4. By taking the principal normal and binormal vectors of the $X-pedal$ curve as position vectors, we define a regular curve called the α_3- Smarandache curve belonging to $X-pedal$ curve obtained from these vectors

$$\alpha_3 = \frac{N_1 + B_1}{\sqrt{2}} \tag{2.26}$$

Theorem II.8. If $T_{\alpha_3}, N_{\alpha_3}$ and B_{α_3} denote the Frenet vectors of α_3- Smarandache curve, then these vectors are computed as follows.

$$T_{\alpha_3} = \frac{-\kappa_1 T_1 - \tau_1 N_1 + \tau_1 B_1}{\sqrt{\kappa_1^2 + 2\tau_1^2}},$$

$$N_{\alpha_3} = B_{\alpha_3} \wedge T_{\alpha_3},$$

$$B_{\alpha_3} = \frac{-(\tau_1 y_3 + \tau_1 y_2)T_1 + (\kappa_1 y_3 + \tau_1 y_1)N_1 + (\tau_1 y_1 - \kappa_1 y_2)B_1}{\sqrt{(\tau_1 y_3 + \tau_1 y_2)^2 + (\kappa_1 y_3 + \tau_1 y_1)^2 + (\tau_1 y_1 - \kappa_1 y_2)^2}}$$

where y_1, y_2 and y_3,

$$y_1 = \frac{\mu_1^2\tau_1\kappa_1 - (\mu_1\kappa_1)'}{\sqrt{2}}, y_2 = -\frac{\mu_1^2(\kappa_1^2 + \tau_1^2) + (\mu_1\tau_1)'}{\sqrt{2}}, y_3 = \frac{(\mu_1\tau_1)' - \mu_1^2\tau_1^2}{\sqrt{2}}.$$

Proof. Let take the derivative of (2.26) and then, take into the account of (2.9). So we take,

$$\alpha_3' = \frac{\mu_1(-\kappa_1 T_1 - \tau_1 N_1 + \tau_1 B_1)}{\sqrt{2}} \tag{2.28}$$

$$\alpha_3'' = y_1 T_1 + y_2 N_1 + y_3 B_1 \tag{2.29}$$

$$\alpha_3''' = \left(y_1' - \mu_1 y_2 \kappa_1\right) T_1 + \left(y_2' + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1\right) N_1 + \left(y_3' + \mu_1 y_2 \tau_1\right) B_1 \qquad (2.30)$$

$$\alpha_3' \wedge \alpha_3'' = \frac{\mu_1}{\sqrt{2}}\left(\left(-\left(\tau_1 y_3 + \tau_1 y_2\right) T_1 + \left(\kappa_1 y_3 + \tau_1 y_1\right) N_1 + \left(\tau_1 y_1 - \kappa_1 y_2\right) B_1\right)\right) \qquad (2.31)$$

$$\det\left(\alpha_3', \alpha_3'', \alpha_3'''\right) = \frac{\mu_1}{\sqrt{2}}\begin{pmatrix}\left(\tau_1 y_1 - \kappa_1 y_2\right)\left(y_3' + \mu_1 y_2 \tau_1\right) - \left(\tau_1 y_3 + \tau_1 y_2\right)\left(y_1' - \mu_1 y_2 \kappa_1\right) \\ + \left(y_1' - \mu_1 y_2 \kappa_1\right)\left(y_2' + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1\right)\end{pmatrix} \qquad (2.32)$$

$$\left\|\alpha_3'\right\| = \frac{\mu_1}{\sqrt{2}}\sqrt{\kappa_1^2 + 2\tau_1^2} \qquad (2.33)$$

$$\left\|\alpha_3' \wedge \alpha_3''\right\| = \frac{\mu_1}{\sqrt{2}}\sqrt{\left(\tau_1 y_3 + \tau_1 y_2\right)^2 + \left(\kappa_1 y_3 + \tau_1 y_1\right)^2 + \left(\tau_1 y_1 - \kappa_1 y_2\right)^2} \qquad (2.34)$$

If substituting the eqs. (2.27), (2.30), (2.32) and (2.33) into (1.2), it completes the proof.

Theorem II.8. Let κ_{α_3} and τ_{α_3} be the curvature and torsion of α_3- Smarandache curve, respectively. These curvatures are obtained as the following equalities.

$$\kappa_{\alpha_3} = \frac{2\sqrt{\left(\tau_1 y_3 + \tau_1 y_2\right)^2 + \left(\kappa_1 y_3 + \tau_1 y_1\right)^2 + \left(\tau_1 y_1 - \kappa_1 y_2\right)^2}}{\mu_1^2\left(\kappa_1^2 + 2\tau_1^2\right)\sqrt{\kappa_1^2 + 2\tau_1^2}},$$

$$\tau_{\alpha_3} = \frac{\sqrt{2}\begin{pmatrix}\left(\tau_1 y_1 - \kappa_1 y_2\right)\left(y_3' + \mu_1 y_2 \tau_1\right) - \left(\tau_1 y_3 + \tau_1 y_2\right)\left(y_1' - \mu_1 y_2 \kappa_1\right) \\ + \left(y_1' - \mu_1 y_2 \kappa_1\right)\left(y_2' + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1\right)\end{pmatrix}}{\mu_1\left(\left(\tau_1 y_3 + \tau_1 y_2\right)^2 + \left(\kappa_1 y_3 + \tau_1 y_1\right)^2 + \left(\tau_1 y_1 - \kappa_1 y_2\right)^2\right)}.$$

Proof. Substituting the expressions (2.31), (2.32) and (2.33) into the relation (1.2), this completes the proof.

Definiton II.5. By taking the tangent, the principal normal and binormal vectors of the $X-pedal$ curve as position vectors, we define a regular curve called the α_4- Smarandache curve belonging to the $X-pedal$ curve obtained from these vectors.

$$\alpha_4 = \frac{T_1 + N_1 + B_1}{\sqrt{3}} \qquad (2.34)$$

Theorem II.9. If the Frenet vectors of the α_4- Smarandache curve are shown as $T_{\alpha_4}, N_{\alpha_4}$ and B_{α_4} then these vectors are calculated as follows.

$$T_{\alpha_4}=\frac{-\kappa_1 T_1+(\kappa_1-\tau_1)N_1+\tau_1 B_1}{\sqrt{2\kappa_1^2-2\kappa_1\tau_1+2\tau_1^2}},$$

$$N_{\alpha_4}=B_{\alpha_4}\wedge T_{\alpha_4},$$

$$B_{\alpha_4}=\frac{(z_3\kappa_1-z_3\tau_1-\tau_1z_2)T_1+(\tau_1z_1+\kappa_1z_3)N_1-(\kappa_1z_1-\tau_1z_1+\kappa_1z_2)B_1}{\sqrt{(z_3\kappa_1-z_3\tau_1-\tau_1z_2)^2+(\tau_1z_1+\kappa_1z_3)^2+(\kappa_1z_1-\tau_1z_1+\kappa_1z_2)^2}}$$

where the coefficients z_1, z_2 and z_3,

$$z_1=-\frac{(\mu_1\kappa_1)'+\mu_1^2\kappa_1(\kappa_1-\tau_1)}{\sqrt{3}}, z_2=\frac{(\mu_1\kappa_1-\mu_1\tau_1)'-\mu_1^2(\kappa_1^2+\tau_1^2)}{\sqrt{3}}, z_3=\frac{(\mu_1\tau_1)'+\mu_1^2\tau_1(\kappa_1-\tau_1)}{\sqrt{3}}.$$

Proof. By taking the derivative of the expression (2.34) and then, taking into account the relation (2.9), following equalities are obtained.

$$\alpha_4'=\frac{\mu_1(-\kappa_1T_1+(\kappa_1-\tau_1)N_1+\tau_1B_1)}{\sqrt{3}} \tag{2.35}$$

$$\alpha_4''=z_1T_1+z_2N_1+z_3B_1 \tag{2.36}$$

$$\alpha_4'''=(z_1'-z_2\mu_1\kappa_1)T_1+(z_2'+z_1\mu_1\kappa_1-z_3\mu_1\tau_1)N_1+(z_3'+z_2\mu_1\tau_1)B_1 \tag{2.37}$$

$$\alpha_4'\wedge\alpha_4''=\frac{\mu_1}{\sqrt{3}}\left((z_3\kappa_1-z_3\tau_1-\tau_1z_2)T_1+(\tau_1z_1+\kappa_1z_3)N_1-(\kappa_1z_1-\tau_1z_1+\kappa_1z_2)B_1\right) \tag{2.38}$$

$$\det(\alpha_4',\alpha_4'',\alpha_4''')=\frac{\mu_1}{\sqrt{3}}\begin{pmatrix}(z_3\kappa_1-z_3\tau_1-\tau_1z_2)(z_1'-z_2\mu_1\kappa_1)+(\tau_1z_1+\kappa_1z_3)(z_2'+z_1\mu_1\kappa_1-z_3\mu_1\tau_1)\\-(\kappa_1z_1-\tau_1z_1+\kappa_1z_2)(z_3'+z_2\mu_1\tau_1)\end{pmatrix} \tag{2.39}$$

$$\|\alpha_4'\|=\frac{\sqrt{6}\mu_1}{3}\sqrt{\kappa_1^2-\kappa_1\tau_1+\tau_1^2} \tag{2.40}$$

$$\|\alpha_4'\wedge\alpha_4''\|=\frac{\mu_1}{\sqrt{3}}\sqrt{(z_3\kappa_1-z_3\tau_1-\tau_1z_2)^2+(\tau_1z_1+\kappa_1z_3)^2+(\kappa_1z_1-\tau_1z_1+\kappa_1z_2)^2} \tag{2.41}$$

By substituting relations (2.35), (2.38), (2.40) and (2.41) into (1.1), the proof is completed.

Theorem II.10 Let κ_{α_4} and τ_{α_4} be the curvature and torsion of α_4- Smarandache curve, respectively. These curvatures are given by the following equalities.

$$\kappa_{\alpha_4}=\frac{3\sqrt{2}}{4}\cdot\frac{\sqrt{\left(z_3\kappa_1-z_3\tau_1-\tau_1z_2\right)^2+\left(\tau_1z_1+\kappa_1z_3\right)^2+\left(\kappa_1z_1-\tau_1z_1+\kappa_1z_2\right)^2}}{\mu_1^2\left(\kappa_1^2-\kappa_1\tau_1+\tau_1^2\right)\sqrt{\kappa_1^2-\kappa_1\tau_1+\tau_1^2}},$$

$$\tau_{\alpha_4}=\frac{\sqrt{3}\begin{pmatrix}\left(z_3\kappa_1-z_3\tau_1-\tau_1z_2\right)\left(z_1'-z_2\mu_1\kappa_1\right)+\left(\tau_1z_1+\kappa_1z_3\right)\left(z_2'+z_1\mu_1\kappa_1-z_3\mu_1\tau_1\right)\\-\left(\kappa_1z_1-\tau_1z_1+\kappa_1z_2\right)\left(z_3'+z_2\mu\tau_1\right)\end{pmatrix}}{\mu_1\left(\left(z_3\kappa_1-z_3\tau_1-\tau_1z_2\right)^2+\left(\tau_1z_1+\kappa_1z_3\right)^2+\left(\kappa_1z_1-\tau_1z_1+\kappa_1z_2\right)^2\right)}.$$

Proof. By substituting relations (2.39), (2.40) and (2.41) into (1.2), the proof is completed.

III.CONCLUSION

In this study, the pedal curve of a unit vector lying on the osculator plane of a curve was defined and the Frenet vectors, curvature and torsion were calculated. Then, Smarandache curves were described by taking Frenet vectors of these pedal curves as position vectors. Finally, Frenet vectors, curvature and torsion of the obtained curves were expressed in terms of the main curve.

ACKNOWLEDGMENT

The authors declare no conflict of interest in this paper.

REFERENCES

[1] A.T. Ali, Special Smarandache Curves in the Euclidean Space, International Journal of MathematicalCombinatorics vol. 2, pp. 30--36, 2010.

[2] A. Gray, E. Abbena and S. Salamon, Modern Differential Geometry of Curves and Surfaces with Mathematica, Chapman and Hall/CRC: Boca Raton, FL, USA, 2016.

[3] A.Y.Ceylan, M.Kara, On Pedal and Contrapedal Curves of Bezier Curves, Konuralp Journal of Mathematics, 9 (2), 217-221, 2021.

[4] E.As, A.Sarioğlugil, (2015). On the Pedal Surfaces of 2-d Surfaces with the Constant Support Function in E^4, Pure Mathematical Sciences, 4(3), 105 – 120, 2015.

[5] M. P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.

[6] M. Özdemir, Diferansiyel Geometri, Altın Nokta Yayınevi, İzmir, 2020.

[7] M. Turgut, S. Yılmaz, Smarandache Curves in Minkowski Space-Time, International J. Math. Combin.,vol. 3, pp. 51--55, 2008.

[8] N.Kuruoğlu,A, Sarıoğlugil, On the characteristic properties of the a-Pedal surfaces in the euclidean space E3. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 42, 19-25, 1993.

[9] O.O. Tuncer, İ. Gök, F. N. Ekmekci, Notes on pedal and contrapedal curves of fronts in the Euclidean plane, Mathematical Methods in the Applied Sciences, 41(13), 5096–5111, 2018.

[10] Y. Li, Donghe Pei, Pedal curves of fronts in the sphere, Journal of Nonlinear Sciences and Applications, 9(3),836—844,2016.

[11] Y. Li, Pei D., Pedal curves of frontals in the Euclidean plane, Math *Meth Appl Sci*., 1-10, 2018.

[12] Y. Li, O. Oğulcan Tuncer, On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space, Mathematical Methods in the Applied Sciences, 46(9),11157-11171, 2023.

[13] S. Şenyurt, S. Sivas, An Application of Smarandache Curve, Ordu University Journal of Science and Tecnology, vol. 3, no. 1, pp. 46--60, 2013.

[14] S. Şenyurt, D. Canlı, and K.H. Ayvacı, Associated Curves From A Different Point Of View In E^3, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat. Volume 71, Number 3, Pages 826–845, (2022).

[15] S. Şenyurt, F. Ertem Kaya, D. Canlı, Pedal curves obtained from Frenet vector of a space curve and Smarandache curves belonging to these curves, AIMS Mathematics, 9(8), 20136-20162, 2024.

[16] F. Ertem Kaya, Differential Geometric Aspects of Pedal Curves on Surfaces, Journal of Mathematics, 1-11, 2024.

[17] D. Canlı, S. Şenyurt, F. Ertem Kaya, L. Grilli, Pedal Curves Generated by Alternative Frame Vectors and Their Smarandache Curves, Symmetry, 16(8), 1-28, 2024.

ALL SCIENCES ACADEMY

Proceeding Book of 2nd International Conference on Pioneer and Innovative Studies ICPIS 2025

January 30 - 31, 2025: Konya, Turkey

PROCEEDING BOOK OF 2ND INTERNATIONAL CONFERENCE ON PIONEER AND INNOVATIVE STUDIES ICPIS 2025

PROCEEDING BOOK OF 2ND ICPIS 2025:

30-31 January 2025

Konya, Turkey

Publication date: 06.02.2025

Publisher: All Sciences Academy

Proceeding Book of 2nd International Conference on Pioneer and Innovative Studies ICPIS 2025

https://as-proceeding.com/index.php/icpis/home

2nd International Conference on Pioneer and Innovative Studies ICPIS 2025: edited by All Sciences Academy.

ISBN: 978-625-5954-58-9

COMMITTEE

ORGANIZING COMMITTEES

Assoc. Prof. Admir Jance, European University of Tirana
Assoc. Prof. Elif Küçük Durur, Atatürk University
Assoc. Prof. Nermin Kişi, Zonguldak Bülent Ecevit University
Assoc. Prof. Ülhak Çimen, Atatürk University
Dr. Krisztina Czakoova, J. Selye University
Dr. Ondrej Takac, J. Selye University
Dr. Ladislav Vegh, J. Selye University

SCIENTIFIC COMMITTES

Prof. Dr. Abdurazzag Ali Aburas, University of Kwazulu-Natal
Prof. Dr. Adnan Sözen, Gazi University
Prof. Dr. Amit Soni, Manıpal University
Prof. Dr. Anis Koubaa, Prince Sultan University
Prof. Dr. Carlos A. Coelho, Universidade Nova de Lisboa
Prof. Dr. Josep Guerrero, Aalborg University
Prof. Dr. Juan Vasquez, Aalborg University
Prof. Dr. Laszlo T. Koczy, Budapest University of Technology and Economics
Prof. Dr. Mohammad Ashraf Gondal, King Fahd University
Prof. Dr. Murat Yakar, Mersin University, Turkey
Prof. Dr. Muhammad Rashid, University of West Florida
Prof. Dr. Saad Mekhilef, Malaya University
Prof. Dr. Salim Ibrir, King Fahd University
Prof. Dr. Subarna Shakya, Tribhuvan University
Prof. Dr. Suzuki Hisashi, Chuo University
Prof. Dr. Usha Mehta, Institute of Technology, Nirma University
Prof. Dr. Vagif Gasimov, Azerbaijan Technical University
Prof. Dr. Weidong Zhu, University of Maryland
Assoc. Prof. Ali Hakan Işık, Burdur Mehmet Akif Ersoy University
Assoc. Prof. Amita Nandal, Manıpal University
Assoc. Prof. Aslan İnan, Yıldız Teknik University
Assoc. Prof. Atif Iqbal, Qatar University
Assoc. Prof. Elif Küçük Durur, Atatürk University
Assoc. Prof. Francesco Cottone, University of Perugia
Assoc. Prof. Farid Melgani, University of Trento
Assoc. Prof. Halit Öztekin, Sakarya Applied Science University
Assoc. Prof. Hamdy Ziedan, Assiut University
Assoc. Prof. Harun Kemal Öztürk, Pamukkale University
Assoc. Prof. Jiri Pela, Brno University of Technology
Assoc. Prof. Merih Palandöken, İzmir Katip Çelebi University
Assoc. Prof. Nermin Kişi, Zonguldak Bülent Ecevit University
Assoc. Prof. Rita Ismailova, Kyrgyz Turkish Manas University
Assoc. Prof. Sadia Samar Ali, King Abdul-Aziz University
Assoc. Prof. Ülhak Çimen, Atatürk University
Assoc. Prof. Tanmay De, National Institute of Technology Durgapur
Assoc. Prof. Zeashan Khan, Air University
Dr. Ali Almisreb, International University of Sarajevo
Dr. Amina Othmani, University of Monastir-Tunisia
Dr. Arti Jain, Jaypee Institute of Information Technology
Dr. Arun Kumar Verma, Malaviya National Institute of Technology Jaipur
Dr. Aziz Shah, University of Glaskow
Dr. Bal Chand Nagar, ECE National Institute of Technology Patna
Dr. Baoze Wei, Aalborg University
Dr. Bohumil Brtník, University of Pardubice
Dr. Diptiranjan Behera, The University of the West Indies at Mona
Dr. Fayçal BENSAALI, Qatar University
Dr. Fugui Qi, Fourth Military Medical University
Dr. Magdi S. Mahmoud, King Fahd University
Dr. Mariya Ouaissa, Moulay Ismail University
Dr. Merdaci Slimane, University of -Sidi Bel Abbés, Algeria
Dr. Moharram Challenger, Antwerp University
Dr. Muhammad Sultan, Zakariya University
Dr. Lesya Anishchenko, Bauman Moscow State Technical University
Dr. R.A. Ilyas, Universiti Teknologi Malaysia
Dr. Sadina Gagula Palalić, International University of Sarajevo
Dr. Stefania Tomasiello, University of Tartu
Dr. Syed Omer Gilani, National University of Sciences and Technology (NUST)
Dr. Tarık Namas, International University of Sarajevo
Dr. Uduak A. Umoh, University of Uyo
Dr. Yajuan Guan, Aalborg University